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The maximum potential energy of elastic rods under axial gravitational load

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Abstract

An optimal design problem solved previously for an elastic rod hanging under its own weight found the distribution of the cross-sectional area that minimized the total potential energy stored in an equilibrium state, with the admissible designs bounded above and below and also subject to the constraint of prescribed total volume. This work solves the companion problem of the design that stores the maximum potential energy under the same constraint conditions. The method used is based on a comparison theorem for sandwich structures.

Keywords: Optimal design; Elastic rods; Sandwich structures

1. The maximum energy design problem

The problem solved here is the determination of the distribution of the cross-sectional area $A(X)$ and associated axial displacement $U(X)$ which maximizes the total equilibrium potential energy of an elastic rod hanging vertically under its own weight:

$$\max_A(\min_U(\Pi(U;A))) \tag{1a}$$

where

$$\Pi(U;A) = \int_0^L \left\{ \frac{1}{2}EA(X)\varepsilon^2(X) - \rho g A U(X) \right\} \mathrm{d}X \tag{1b}$$

with the axial strain $\varepsilon(X) = \mathrm{d}U/\mathrm{d}X$. The admissible designs for cross-sectional area are positive, piecewise continuous functions, bounded from above and below and satisfying the integral constraint of prescribed total volume:

$$0 < A_1 \leqslant A(X) \leqslant A_2; \qquad \int_0^L A(X)\mathrm{d}X = L\overline{A}, \quad A_1 < \overline{A} < A_2 \tag{1c}$$

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Kinematically admissible axial displacement functions $U(X)$ are continuous functions with piecewise continuous first derivatives satisfying displacement boundary conditions

$$U(0) = 0, \quad U(L) = \Delta \tag{1d}$$

This problem is a companion to one solved earlier (Warner, 2000, 2001):

$$\min_A(\min_U(\Pi(U;A)))$$

under the same conditions on A and U. A reviewer of one of these papers suggested that it might be more interesting physically to find the design that stored the maximum rather than the minimum potential energy in the rod. This paper shows how that question can be answered. The method is not the Pontryagin Maximum Principle used in Warner (2000, 2001) but is based on a comparison theorem for sandwich structures that characterizes an optimal design if one exists. I note that the "min–min" problem was solved essentially first by Fosdick and Royer-Carfagni (1996) in the context of mixture theory with a concentration density rather than area as the design variable; their solution was found by a different method than mine.

The elasticity problem has solution that can be characterized by the Principle of Minimum Potential Energy for each given area distribution: among all kinematically admissible displacement functions, the solution displacement (which is unique) makes the functional Π an absolute minimum. The Euler–Lagrange equation for this variational principle is the equilibrium equation on force per unit length written as a second-order differential equation on U:

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(EA\frac{\mathrm{d}U}{\mathrm{d}X}\right) + \rho g A = 0 \tag{2a}$$

Introduction of the (continuous) axial force function $P(X)$ and the elastic law $P = EA\varepsilon$ allows us to write this last equation in the form

$$\frac{\mathrm{d}P}{\mathrm{d}X} = \frac{\mathrm{d}}{\mathrm{d}X}(EA\varepsilon) = -\rho g A \tag{2b}$$

Three conclusions which will be useful later follow. First, from the elastic law, since P must be continuous, any discontinuity in A must occur where ε is also discontinuous unless $P = \varepsilon = 0$ at the point of discontinuity. Second, since $A > 0$, P is monotone decreasing and so can have at most one zero. That is, the rod can be all in tension or all in compression or may be partly in tension and partly in compression with the tension side at the top of the rod next to the end $X = 0$. Lastly, since $A(X)$ occurs homogeneously in the equilibrium equation, $A(X)$ can be scaled by any multiplicative factor without changing the solution for $U(X)$. The value of the potential energy functional Π is then subject to the same scaling as A.

A nondimensional form of the problem is stated next. The comparison theorem is then derived. Arguments based on the continuity of the axial force show that a design satisfying the comparison theorem must be continuous and must consist of one-to-five subdesigns in a particular order, for a total of 12 possible cases depending on the parameters of the problem. A master area function and its associated axial displacement are constructed from which the equations for all 12 cases can be derived. The nature of the solution to these equations is discussed with those that can be solved in closed form presented explicitly.

The solution procedure for the problem when no upper bound on section size is prescribed will be summarized. The solution for the hanging rod with its lower end free will also be discussed.

2. Nondimensionalization—the comparison lemma for the rod

The rod problem falls into the class of optimal design problems for sandwich structures: those for which the elastic stiffness (here EA) and the body force or load coefficient (ρgA) are both linear in the design variable (A). Construction of the solutions is based on a comparison theorem for sandwich structures of a type derived by Prager and Taylor (1968) without upper and lower bounds on the design variable and by Prager (1971) with a lower bound. The latter was used by Cardou and Warner (1974) to develop a design rule for framed structures vibrating at a given frequency. Here the theorem characterizing globally optimal solutions is proven with both bounds present and applied to the rod problem.

It is best to nondimensionalize the problem in order to keep the notation as simple as possible in the analysis. Change from $\{X, A, U, P, \varepsilon, \Pi\}$ to $\{x, a, u, p, \eta, \pi\}$ by setting

$$\begin{aligned} &X = Lx; \quad A(X) = \widehat{A}a(x); \quad U(X) = \frac{\rho g L^2}{E} u(x) \\ &P(X) = \rho g L \widehat{A} p(x); \quad \varepsilon(X) = \frac{\rho g L}{E} \eta(x) \\ &\Pi(U; A) = \frac{\rho^2 g^2 L^3 \widehat{A}}{E} \pi(u; a) \end{aligned} \tag{3}$$

Because of the scaling property on $A(X)$ mentioned in Section 1, the choice of the reference area $\widehat{A}$ is arbitrary, with A_1 or A_2 natural choices. In Warner (2000) (with a similar construction occurring in Fosdick and Royer-Carfagni (1996)) it was taken for the min–min problem as a weighted harmonic mean of the bounding values with the weighting ratios dependent on all three of A_1, A_2 and $\bar{A}$. I shall not make a choice, however, but leave it unspecified so that the dependence of the results on the ratios of the area parameters is clear. Comparison of the energy values here to those in Warner (2000) will require accounting for the area scaling.

In this process, the nondimensional ratio $\rho g L/E$ has been incorporated in the changes of variables rather than left as a parameter in the statement of the problem. The value of this ratio for a rod five meters long made of steel with $\rho g = 77$ kN/m and $E = 200$ GPa is about 2×10^{-6}. The value of the nondimensional end displacement δ will then equal about 5000 for an actual elongation $\Delta = 0.01L$ and about 500 for $\Delta = 0.001L$. Thus $|\delta|$ can be "large"—even a "compression" apparently much, much greater than the "undeformed length" of 1.

In the new notation the problem becomes that of finding $a(x)$ and $u(x)$ such that they solve

$$\max_a \left[\min_u \left(\pi(u; a) \equiv \int_0^1 a(x) \lambda(x) \, \mathrm{d}x \right) \right] \tag{4a}$$

with the energy density function

$$\lambda(x) = \frac{1}{2} \eta^2(x) - u(x) \tag{4b}$$

and

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \eta, \quad u(0) = 0, \quad u(1) = \delta = \frac{E}{\rho g L^2} \Delta \tag{4c}$$

$$\begin{aligned} &0 < a_1 \leqslant a(x) \leqslant a_2; \\ &\int_0^1 a(x) \mathrm{d}x = \bar{a}, a_1 < \bar{a} < a_2 \end{aligned} \tag{4d}$$

Here $(a_1, \bar{a}, a_2) = (A_1, \bar{A}, A_2)/\widehat{A}$. The equilibrium equation becomes

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(a\eta) = \frac{\mathrm{d}}{\mathrm{d}x}\left(a \frac{\mathrm{d}u}{\mathrm{d}x} \right) = -a \tag{5}$$

The derivation of the comparison theorem follows. Suppose $u^*(x)$ solves the elasticity problem for the design $a^*(x)$ and $u^{**}(x)$ solves it for $a^{**}(x)$. Then the difference between the two actual solution values of the potential energy can be written and transformed as follows:

$$\begin{aligned}\pi(u^*;a^*) - \pi(u^{**};a^{**}) &= \int_0^1 \{a^*\lambda^* - a^{**}\lambda^{**}\}\mathrm{d}x = \int_0^1 a^{**}\{\lambda^* - \lambda^{**}\}\mathrm{d}x + \int_0^1 \{a^* - a^{**}\}\lambda^*\mathrm{d}x \\ &= \pi(u^*;a^{**}) - \pi(u^{**};a^{**}) + \lambda_0\int_0^1 \{a^* - a^{**}\}\mathrm{d}x + \int_0^1 \{a^* - a^{**}\}\{\lambda^* - \lambda_0\}\mathrm{d}x\end{aligned}$$

Here λ_0 is any number. The first two terms taken together are positive by the Principle of Minimum Potential Energy for the design a^{**} since u^* is admissible in that variational competition. The next integral will be zero if both a^* and a^{**} satisfy the fixed-volume constraint. In the last integral, the first factor must be nonnegative for all admissible a^{**} wherever $a^* = a_2$ and nonpositive wherever $a^* = a_1$. Thus we have the following theorem.

Comparison Theorem. *Suppose that, for an admissible design $a^*(x)$ and its associated elastic solution functions, there exists a number λ_0 such that*

$$\begin{aligned}&\lambda^*(x) - \lambda_0 \geqslant 0 \quad \textit{wherever } a^*(x) = a_2\\ &\lambda^*(x) - \lambda_0 \leqslant 0 \quad \textit{wherever } a^*(x) = a_1\\ &\lambda^*(x) - \lambda_0 = 0 \quad \textit{wherever } a_1 < a^*(x) < a_2\end{aligned}$$

Then the last integral as well as the first two terms in the difference of the potential energies will be nonnegative for all other admissible designs $a^{**}(x)$ and so $a^*(x)$ will be a solution of the maximum stored potential energy problem.

Designs $a(x)$ that satisfy these conditions are easy to construct. They must be continuous and are made up of standard pieces joined together in which either $a(x)$ is at one of its bounds or else the associated energy density $\lambda(x)$ is a constant λ_0. There are 12 cases in all, of which seven must be considered in full detail; the other five can be found from the solutions of members of the first seven by changing the sign of the prescribed deflection and appropriately interchanging values of the solution terms. The best way of organizing the information that I have found is not by constructing each type independently but by constructing a ''master'' area function and its associated displacement covering all possibilities without regard to the displacement boundary conditions or the integral constraint and then showing how each case can be obtained by translating and scaling the independent variable in the master functions to put the endpoints of the rod at their proper positions so that the boundary conditions are satisfied and the volume constraint is met.

3. Continuity of comparison theorem designs

As stated earlier, the area $a(x)$ and the strain $\eta(x)$ must be continuous or simultaneously discontinuous except possibly for a case where p and η vanish at a discontinuity in a. This last possibility is ruled out by the construction given later. That a discontinuity cannot occur where p is not zero in a comparison theorem design follows from the fact that the product of a and η must be continuous at such a point and so the strain on the side of the join with larger a value must be less than the strain on the side with smaller a. But then the value of $\lambda(x)$ on the side with larger a would also be less than that on the side with smaller a and so violate the comparison theorem conditions on the sign of $\lambda - \lambda_0$ in the different regions.

Thus $a(x)$ must be continuous unless the exceptional case of $p = 0$ at the jump occurs. Even if this were possible, however, note that $\eta(x)$ must be zero there and so η and λ would be continuous even if a were not.

4. Solution forms in regions where $a(x)$ or $\lambda(x)$ is constant

In an interval where $a(x)$ is any constant, the equilibrium equation becomes $u'' = -1$ so that the displacement, strain, and energy density have the forms

$$\begin{cases} u(x) = -\dfrac{1}{2}(x-\hat{x})^2 + \hat{\eta}(x-\hat{x}) + \hat{u} \\ \eta(x) = \hat{\eta} - (x-\hat{x}) \\ \lambda(x) = (x-\hat{x})^2 - 2\hat{\eta}(x-\hat{x}) + \dfrac{1}{2}\hat{\eta}^2 - \hat{u} \\ \qquad = \eta^2(x) - \left(\dfrac{1}{2}\hat{\eta}^2 + \hat{u}\right) \end{cases} \tag{6}$$

where $\hat{x}$ is any point of the interval and $\hat{\eta}$, $\hat{u}$ are constants of integration giving the values of the strain and displacement at $\hat{x}$.

In an interval where $\lambda(x)$ has a constant value λ_0, the formula defining λ becomes a differential equation to be solved for $u(x)$. The equilibrium equation then determines $a(x)$. The results are

$$\begin{cases} u(x) = \dfrac{1}{2}(x+\tilde{x})^2 - \lambda_0 \\ \eta = x + \tilde{x} \\ a(x) = \dfrac{\tilde{a}}{(x+\tilde{x})^2} \end{cases} \tag{7}$$

where $\tilde{x}$ and $\tilde{a}$ are constants of integration to be determined along with the proper value for λ_0. The point $x = -\tilde{x}$ cannot lie in the interval if $a(x)$ is to remain bounded.

There are two branches to the curve $a(x)$, one on either side of the point $-\tilde{x}$. In any interval of this type either the branch where $x + \tilde{x} > 0$ or the branch where $x + \tilde{x} < 0$ must be selected. By examining the derivative of $a(x)$ in each interval, one finds that $a(x)$ decreases monotonically in the former case with $\eta(x)$ and $p(x) = \tilde{a}/(x+\tilde{x})$ positive there with the opposite results holding where $x + \tilde{x} < 0$. Thus the rod is in tension in a $x + \tilde{x} > 0$ interval and in compression in a $x + \tilde{x} < 0$ interval.

This means that $p(x)$ can never be zero in a $\lambda = \lambda_0$ interval including its endpoints. Since p is monotonically decreasing, it follows that if both types of λ_0 intervals are present, the tension region must lie above the compression region and be joined to it by a constant section region. This latter must necessarily have $a(x)$ at its lower bound value, since $a(x)$ decreases in the tensile region and increases in the compressive. Moreover, since $\lambda(x)$ is continuous it must be equal to λ_0 at the ends of this minimum section region. It will then be less than λ_0 throughout the region since its graph is a parabola concave upward, one of the conditions for a comparison theorem design. More, p and η must vanish at the midpoint of this region.

It follows that a region at maximum section size must have one end at $x = 0$ or $x = 1$ and the other joined continuously to a λ_0 region. A region at minimum size may also be at either end of the rod joining a λ_0 region at its other end or (as we have just shown) lie entirely within the interval between two λ_0 intervals. A λ_0 interval can occupy all of (0, 1) or be part of a number of different orderings involving two to five regions. Since the whole rod cannot be at maximum or minimum size because of the fixed volume constraint and since we have the monotone nature of $p(x)$ to consider, we can conclude that the only possibilities must be drawn by starting somewhere in the ordering a_2–λ_0 (with decreasing a)–a_1–λ_0 (with increasing a)–a_2

again, with the ending point somewhere later in this sequence. There are 12 such choices possible: two with $\lambda = \lambda_0$ for all x, one totally in tension, the other in compression; four two-region designs, a_2–λ_0, λ_0–a_1, a_1–λ_0 and λ_0–a_2; three with three regions, a_2–λ_0–a_1, λ_0–a_1–λ_0 and a_1–λ_0–a_2; two with four regions, a_2–λ_0–a_1–λ_0 and λ_0–a_1–λ_0–a_2; and one with all five. Though each of these can be constructed directly for x in (0, 1) using continuity of a, η, and λ at the join points together with the displacement boundary conditions and the volume constraint to determine all the constants of integration, the whole computation set can be organized by computing a master area function on the whole line for the ordering of the five possible regions given above and then showing how the rod solutions can be found by imposing the boundary and constraint conditions to show where on the line the $x = 0$ and $x = 1$ points should lie. This construction is given next.

5. The master generating functions for comparison theorem designs; the equations for the 12 cases

I construct a function $a^*(y)$, $-\infty < y < +\infty$, that is bounded above and below: $a_1 \leqslant a^*(y) \leqslant a_2$ even about $y = 0$ which is taken as the midpoint of the region at minimum section size a_1 and consists of five sections in the order discussed above. If the endpoints of this middle section at minimum size are taken at $y = \pm 1$, then it is an exercise in algebra from the continuity conditions at either end of the λ_0 regions to show that the other ends of those regions must lie at $y = \pm(2 - \alpha)$, $\alpha = \sqrt{(a_1/a_2)} < 1$, and to determine the other constants of integration (except for the value to pick for λ_0):

$$a^*(y) = \begin{cases} a_2, & y \leqslant -(2-\alpha) \\ a_1/(2+y)^2, & -(2-\alpha) \leqslant y \leqslant -1 \\ a_1, & -1 \leqslant y \leqslant 1 \\ a_1/(2-y)^2, & 1 \leqslant y \leqslant 2-\alpha \\ a_2, & 2-\alpha \leqslant y \end{cases} \tag{8}$$

The corresponding displacement, strain, and Lagrangian density functions are

$$u^*(y) + \lambda_0^* = \begin{cases} \alpha^2 - \frac{1}{2}[2(1-\alpha)+y]^2 \\ \frac{1}{2}(2+y)^2 \\ 1 - \frac{1}{2}y^2 \\ \frac{1}{2}(2-y)^2 \\ \alpha^2 - \frac{1}{2}[2(1-\alpha)-y]^2 \end{cases} \tag{9}$$

$$\eta^*(y) = \frac{\mathrm{d}u^*}{\mathrm{d}y} = \begin{cases} -[2(1-\alpha)+y] \\ 2+y \\ -y \\ y-2 \\ 2(1-\alpha)-y \end{cases} \tag{10}$$

$$\lambda^*(y) - \lambda_0^* = \frac{1}{2}\eta^{*2} - (u^* + \lambda_0^*) = \begin{cases} [2(1-\alpha)+y]^2 - \alpha^2 \\ 0 \\ y^2 - 1 \\ 0 \\ [2(1-\alpha)-y]^2 - \alpha^2 \end{cases} \tag{11}$$

Graphs of these functions are given in Figs. 1–4. Solutions for the rod are found from these by setting $y = (y_2 - y_1)x + y_1 = \gamma x + y_1$ so that the x-interval (0, 1) maps onto the y-interval (y_1, y_2). The values of the (y_1, y_2) as well as the value of λ_0 are determined by applying the boundary conditions and volume constraint to $a(x) = a^*(\gamma x + y_1)$, $u(x) + \lambda_0 = \gamma^{-2}[u^*(\gamma x + y_1) + \lambda_0^*]$. Expressed in terms of the universal functions, these conditions become

$$\begin{cases} \lambda_0 = u(0) + \lambda_0 = \gamma^{-2}[u^*(y_1) + \lambda_0^*] \\ \delta = \gamma^{-2}[(u^*(y_2) + \lambda_0^*) - (u^*(y_1) + \lambda_0^*)] \\ \gamma \bar{a} = \int_{y_1}^{y_2} a^*(y)\mathrm{d}y \end{cases} \tag{12}$$

The first of these determines λ_0 once the other two have been solved for (y_1, y_2). Table 1 presents the ranges for (y_1, y_2) for each case as well as the form for computing λ_0. Fig. 5 shows the regions of the (y_1, y_2) plane where the values for each case lie.

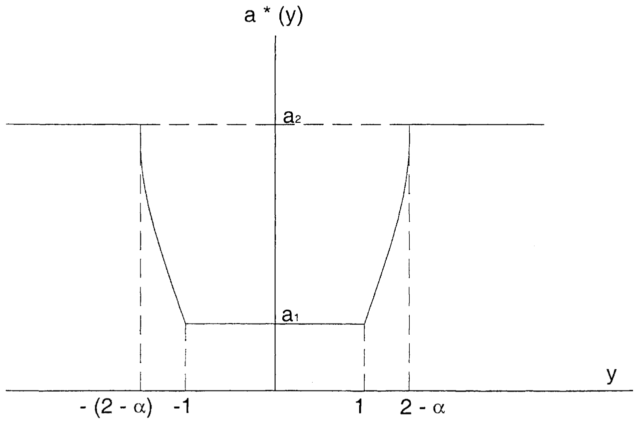

Fig. 1. The master area distribution function $a^*(y)$.

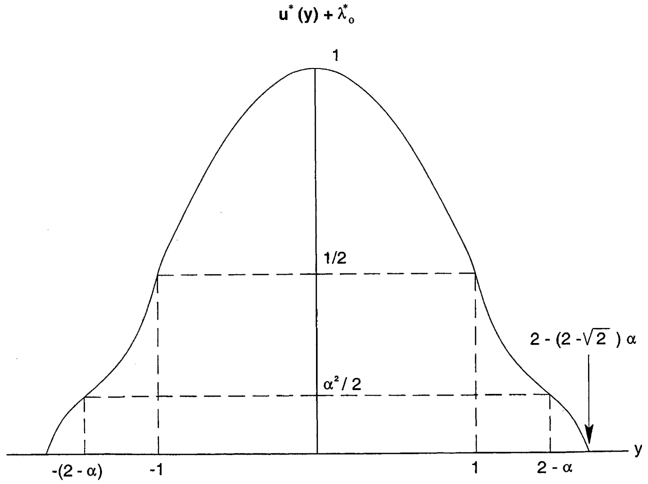

Fig. 2. The master displacement function $u^*(y) + \lambda_0^*$.

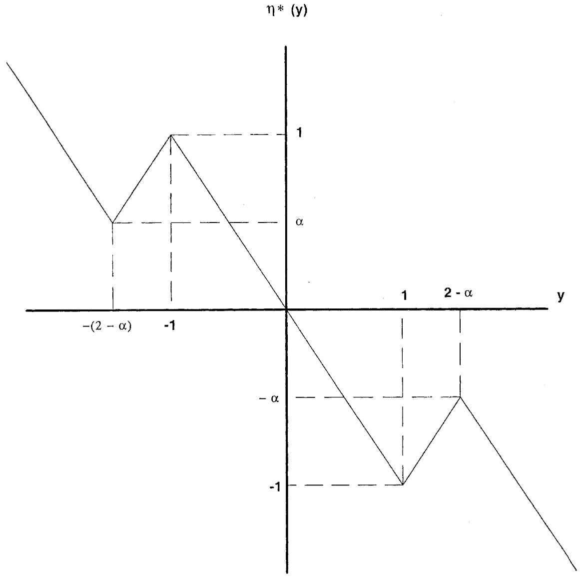

Fig. 3. The master strain function $\eta^*(y)$.

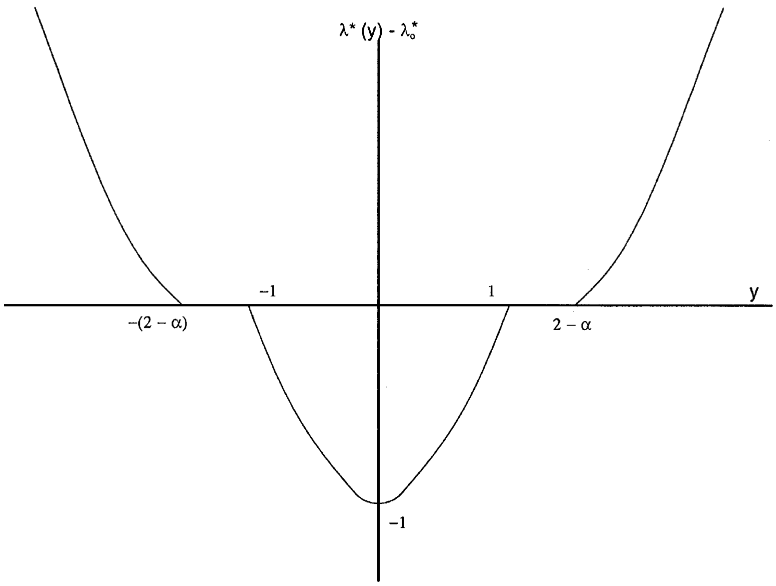

Fig. 4. The master energy density function $\lambda^*(y) - \lambda_0^*$.

The other two equations for each case are listed next, with the first equation coming from the displacement boundary condition at $x = 1$ and the second from the fixed-volume integral constraint. The first involves the parameters δ and α but not $\bar{a}$; the second involves α and $\bar{a}$ but not the extension δ.

Table 1
The (y_1, y_2) ranges and the λ_0 formula for cases (i)–(xii)

Case	y_1 range	y_2 range	$2\gamma^2\lambda_0,\ \gamma = y_2 - y_1$
(i)	$-(2-\alpha) < y_1 < -1$	$-(2-\alpha) < y_2 < -1$	$(2+y_1)^2$
(ii)	$1 < y_1 < (2-\alpha)$	$1 < y_2 < (2-\alpha)$	$(2-y_1)^2$
(iii)	$y_1 < -(2-\alpha)$	$-(2-\alpha) < y_2 < -1$	$2\alpha^2 - [2(1-\alpha)+y_1]^2$
(iv)	$1 < y_1 < (2-\alpha)$	$(2-\alpha) < y_2$	$(2-y_1)^2$
(v)	$-(2-\alpha) < y_1 < -1$	$-1 < y_2 < 1$	$(2+y_1)^2$
(vi)	$-1 < y_1 < 1$	$1 < y_2 < (2-\alpha)$	$2-y_1^2$
(vii)	$y_1 < -(2-\alpha)$	$-1 < y_2 < 1$	$2\alpha^2 - [2(1-\alpha)+y_1]^2$
(viii)	$-1 < y_1 < 1$	$(2-\alpha) < y_2$	$2-y_1^2$
(ix)	$y_1 < -(2-\alpha)$	$1 < y_2 < (2-\alpha)$	$2\alpha^2 - [2(1-\alpha)+y_1]^2$
(x)	$-(2-\alpha) < y_1 < -1$	$(2-\alpha) < y_2$	$(2+y_1)^2$
(xi)	$-(2-\alpha) < y_1 < -1$	$1 < y_2 < (2-\alpha)$	$(2+y_1)^2$
(xii)	$y_1 < -(2-\alpha)$	$(2-\alpha) < y_2$	$2\alpha^2 - [2(1-\alpha)+y_1]^2$

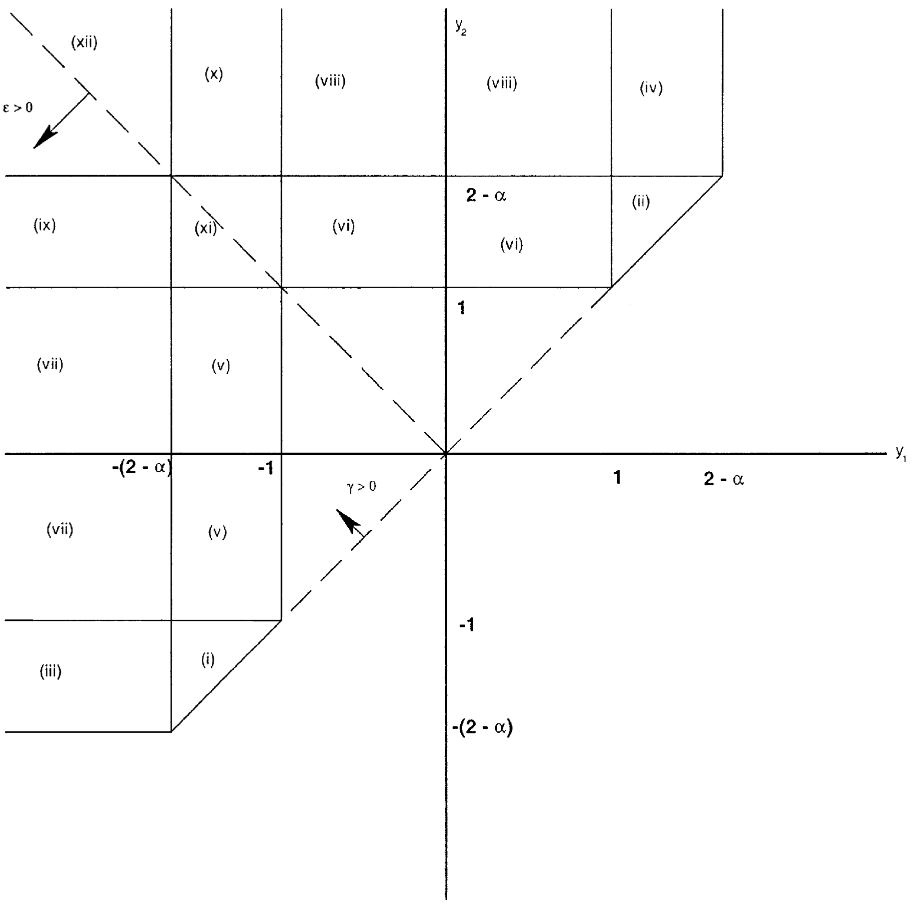

Fig. 5. Regions in the (y_1, y_2) plane where cases (i)–(xii) provide the optimal solution.

Case (i):

$$\begin{cases} 2\gamma^2\delta = (2+y_2)^2 - (2+y_1)^2 \\ \gamma\bar{a} = \int_{y_1}^{y_2} \dfrac{a_1}{(2+y)^2}\,\mathrm{d}y = \dfrac{\gamma a_1}{(2+y_1)(2+y_2)} \end{cases} \tag{13}$$

Case (ii):

$$\begin{cases} 2\gamma^2\delta = (2-y_2)^2 - (2-y_1)^2 \\ \gamma\bar{a} = \int_{y_1}^{y_2} \dfrac{a_1}{(2-y)^2}\,\mathrm{d}y = \dfrac{\gamma a_1}{(2-y_1)(2-y_2)} \end{cases} \tag{14}$$

Case (iii):

$$\begin{cases} 2\gamma^2\delta = (2+y_2)^2 - 2\alpha^2 + [2(1-\alpha)+y_1]^2 \\ \gamma\bar{a} = \int_{y_1}^{-(2-\alpha)} a_2\,\mathrm{d}y + \int_{-(2-\alpha)}^{y_2} \dfrac{a_1}{(2+y)^2}\,\mathrm{d}y \\ \quad = -a_2[2(1-\alpha)+y_1] - \dfrac{a_1}{(2+y_2)} \end{cases} \tag{15}$$

Case (iv):

$$\begin{cases} 2\gamma^2\delta = 2\alpha^2 - [2(1-\alpha)-y_2]^2 - (2-y_1)^2 \\ \gamma\bar{a} = -a_2[2(1-\alpha)-y_2] - \dfrac{a_1}{(2-y_1)} \end{cases} \tag{16}$$

Case (v):

$$\begin{cases} 2\gamma^2\delta = 2 - y_2^2 - (2+y_1)^2 \\ \gamma\bar{a} = a_1 y_2 + \dfrac{a_1}{(2+y_1)} \end{cases} \tag{17}$$

Case (vi):

$$\begin{cases} 2\gamma^2\delta = (2-y_2)^2 - (2-y_1^2) \\ \gamma\bar{a} = -a_1 y_1 + \dfrac{a_1}{(2-y_2)} \end{cases} \tag{18}$$

Case (vii):

$$\begin{cases} 2\gamma^2\delta = (2-y_2^2) - 2\alpha^2 + [2(1-\alpha)+y_1]^2 \\ \gamma\bar{a} = a_1 y_2 - a_2[2(1-\alpha)+y_1] \end{cases} \tag{19}$$

Case (viii):

$$\begin{cases} 2\gamma^2\delta = 2\alpha^2 - [2(1-\alpha)-y_2]^2 - (2-y_1^2) \\ \gamma\bar{a} = -a_1 y_1 - a_2[2(1-\alpha)-y_2] \end{cases} \tag{20}$$

Case (ix):

$$\begin{cases} 2\gamma^2\delta = (2-y_2)^2 - 2\alpha^2 + [2(1-\alpha)+y_1]^2 \\ \gamma\bar{a} = -a_2[2(1-\alpha)+y_1] + \dfrac{a_1}{(2-y_2)} \end{cases} \tag{21}$$

Case (x):

$$\begin{cases} 2\gamma^2\delta = 2\alpha^2 - [2(1-\alpha) - y_2]^2 - (2+y_1)^2 \\ \gamma\bar{a} = -a_2[2(1-\alpha) - y_2] + \dfrac{a_1}{(2+y_1)} \end{cases} \tag{22}$$

Case (xi):

$$\begin{cases} 2\gamma^2\delta = (2-y_2)^2 - (2+y_1)^2 \\ \gamma\bar{a} = \dfrac{a_1}{(2-y_2)} + \dfrac{a_1}{(2+y_1)} \end{cases} \tag{23}$$

Case (xii):

$$\begin{cases} 2\gamma^2\delta = [2(1-\alpha) + y_1]^2 - [2(1-\alpha) - y_2]^2 \\ \gamma\bar{a} = -a_2[2(1-\alpha) - y_2] - a_2[2(1-\alpha) + y_1] \end{cases} \tag{24}$$

Establishing a procedure for the solution of these equations is straightforward. Except for a cubic equation arising from the volume constraint in case (xi), all equations after clearing of fractions are linear or quadratic. A geometric interpretation of the solutions as occurring at the intersection of a conic section (mostly hyperbolas) with another conic or straight line can be made and guide the eventual need for numerical solution in particular cases; cases (i), (ii), and (xii) are solvable in closed form.

Only the odd-numbered cases and case (xii) must be analyzed in detail. The solutions for the even-numbered cases (ii)–(x) can be found from the solutions for the odd-numbered one just above it in the listing. If (y_1, y_2) is the solution to an odd-numbered case (i)–(ix) for given δ and $\bar{a}$, then $(-y_2, -y_1)$ will be a solution to the associated even-numbered case for $-\delta$ and $\bar{a}$; moreover, the end displacement δ will be positive for the odd numbers, always representing extension, so that the even-numbered cases represent compressive states. The same solution property holds for each of (xi) and (xii) individually; all solutions for $\delta = 0$ come from (xi) or (xii). These results follow from showing that the interchange of values and signs transforms the equations of one case to another for (i) to (x) and does not change the equations for (xi) and (xii) except for the sign of δ.

The equations involving δ are quadratic with the one for case (i) reducing to a linear equation once the positive factor $\gamma = y_2 - y_1$ is divided out. The equations for (vii), (xi), and (xii) have hyperbolas as graphs. Those for (iii), (v) and (ix) graph as hyperbolas or ellipses depending upon the value of δ.

The prescribed volume equations also graph as hyperbolas in cases (i), (iii), (v), and (ix) but as straight lines for (vii) and (xii). For (xi) a cubic equation occurs; its graph has symmetry properties that aid in its analysis.

The scaling property of the area function is manifest from these volume equations: only the ratio of the area parameters is needed. We have already introduced α where $\alpha^2 = a_1/a_2$; the ratio $\bar{a}/a_2$ will be called $\bar{v}$ and the equations rewritten using α and $\bar{v}$.

6. The ranges of (δ, $\bar{v}$) where each case is valid and the basic solution procedures

The next step in the analysis is the determination of the ranges of values of δ and $\bar{v}$ for which each case occurs. This information is presented in Fig. 6 by displaying it as regions in the $(\delta, \bar{v})$ plane for $\delta > 0$ where the cases listed in the figure provide the optimal solution; reflection in the $\delta = 0$ axis will provide the regions for $\delta < 0$. The boundaries between the regions are found easily using the information in Fig. 5 about the common y_1 or y_2 value at the boundaries.

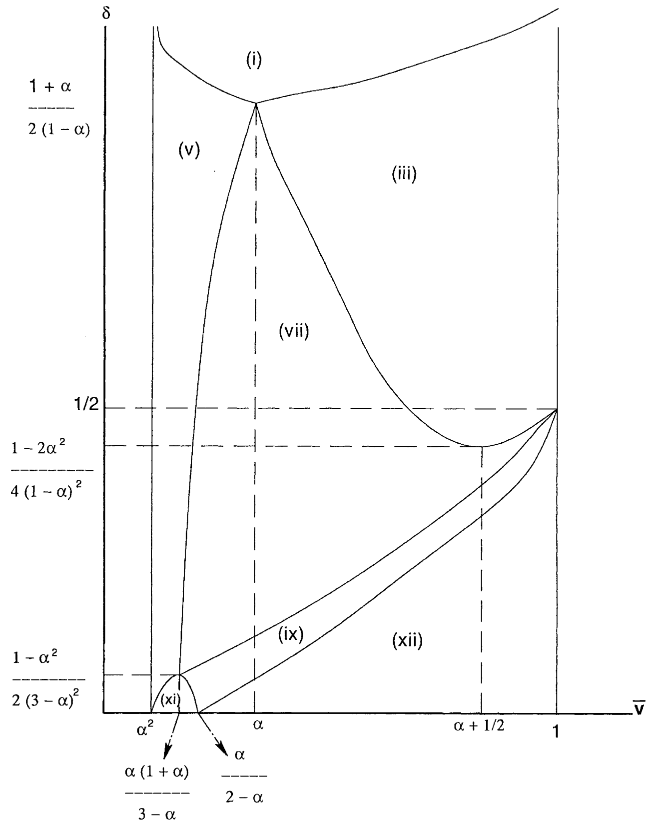

Fig. 6. Regions in the $(\bar{v}, \delta)$ plane for $\delta > 0$ where the seven basic cases provide the optimal solution.

The closed-form solution for case (i) can be found by solving the volume equation for y_1 as a function of y_2 and then substituting in the displacement equation. This is most easily done by introducing new variables (z_1, z_2) equal to $(2 + y_1, 2 + y_2)$ as the natural combinations appearing in the case (i) equations. Indeed all the cases have certain obvious combinations of the y's that should be used to simplify the equations and the discussion of their properties. The transformations to the z's appropriate for each case are given in Table 2, which lists the change of variables from (y_1, y_2) to (z_1, z_2), the ranges of values for (z_1, z_2) and the expression for γ. The corresponding equations for the six odd-numbered cases and case (xii) follow.

The transformed equations for the seven basic cases are (after clearing fractions and canceling common factors):

Case (i):

$$\begin{cases} 2\delta(z_2 - z_1) = z_2 + z_1 \\ \bar{v}z_1 z_2 = \alpha^2 \end{cases} \tag{25}$$

Case (iii):

$$\begin{cases} 2\delta[z_2 - z_1 - 2\alpha]^2 = z_1^2 + z_2^2 - 2\alpha^2 \\ z_2[(1 - \bar{v})z_1 + \bar{v}z_2 - 2\alpha\bar{v}] + \alpha^2 = 0 \end{cases} \tag{26}$$

Table 2
The (z_1, z_2) variables and their ranges for cases (i)–(xii)

Case	z_1	z_2	Ranges of (z_1, z_2)	$\gamma > 0$
(i)	$2 + y_1$	$2 + y_2$	$\alpha < z_1 < z_2 < 1$	$z_2 - z_1$
(ii)	$2 - y_1$	$2 - y_2$	$\alpha < z_2 < z_1 < 1$	$z_1 - z_2$
(iii)	$2(1 - \alpha) + y_1$	$2 + y_2$	$z_1 < -\alpha, \alpha < z_2 < 1$	$z_2 - z_1 - 2\alpha$
(iv)	$2 - y_1$	$2(1 - \alpha) - y_2$	$\alpha < z_1 < 1, z_2 < -\alpha$	$z_1 - z_2 - 2\alpha$
(v)	$2 + y_1$	y_2	$\alpha < z_1 < 1, -1 < z_2 < 1$	$z_2 - z_1 + 2$
(vi)	y_1	$2 - y_2$	$-1 < z_1 < 1, \alpha < z_2 < 1$	$2 - z_2 - z_1$
(vii)	$2(1 - \alpha) + y_1$	y_2	$z_1 < -\alpha, -1 < z_2 < 1$	$z_2 - z_1 + 2(1 - \alpha)$
(viii)	y_1	$2(1 - \alpha) - y_2$	$-1 < z_1 < 1, z_2 < -\alpha$	$2(1 - \alpha) - z_1 - z_2$
(ix)	$2(1 - \alpha) + y_1$	$2 - y_2$	$z_1 < -\alpha, \alpha < z_2 < 1$	$2(2 - \alpha) - z_1 - z_2$
(x)	$2 + y_1$	$2(1 - \alpha) - y_2$	$\alpha < z_1 < 1, z_2 < -\alpha$	$2(2 - \alpha) - z_1 - z_2$
(xi)	$2 + y_1$	$2 - y_2$	$\alpha < z_1 < 1, \alpha < z_2 < 1$	$4 - z_1 - z_2$
(xii)	$2(1 - \alpha) + y_1$	$2(1 - \alpha) - y_2$	$z_1 < -\alpha, z_2 < -\alpha$	$4(1 - \alpha) - z_1 - z_2$

Case (v):

$$\begin{cases} 2\delta[z_2 - z_1 + 2]^2 = 2 - z_1^2 - z_2^2 \\ z_1[\bar{v}z_1 - (\bar{v} - \alpha^2)z_2 - 2\bar{v}] + \alpha^2 = 0 \end{cases} \tag{27}$$

Case (vii):

$$\begin{cases} 2\delta[z_2 - z_1 + 2(1 - \alpha)]^2 = 2(1 - \alpha^2) + z_1^2 - z_2^2 \\ (1 - \bar{v})z_1 + (\bar{v} - \alpha^2)z_2 + 2(1 - \alpha)\bar{v} = 0 \end{cases} \tag{28}$$

Case (ix):

$$\begin{cases} 2\delta[2(2 - \alpha) - z_1 - z_2]^2 = z_1^2 + z_2^2 - 2\alpha^2 \\ z_2[-(1 - \bar{v})z_1 + \bar{v}z_2 - 2(2 - \alpha)\bar{v}] + \alpha^2 = 0 \end{cases} \tag{29}$$

Case (xi):

$$\begin{cases} 2\delta[4 - z_1 - z_2]^2 = z_2^2 - z_1^2 \\ \bar{v}z_1z_2(z_1 + z_2) - 4\bar{v}z_1z_2 + \alpha^2(z_1 + z_2) = 0 \end{cases} \tag{30}$$

Case (xii):

$$\begin{cases} 2\delta[4(1 - \alpha) - z_1 - z_2]^2 = z_1^2 - z_2^2 \\ (1 - \bar{v})(z_1 + z_2) + 4(1 - \alpha)\bar{v} = 0 \end{cases} \tag{31}$$

For (i), substitution of $z_1 = (\alpha^2/\bar{v})z_2^{-1}$ from the $\bar{v}$ equation in the δ equation leads to

$$z_2 = \left(\frac{\alpha}{\sqrt{\bar{v}}}\right)\sqrt{\frac{2\delta + 1}{2\delta - 1}}, \quad z_1 = \left(\frac{\alpha}{\sqrt{\bar{v}}}\right)\sqrt{\frac{2\delta - 1}{2\delta + 1}} \tag{32}$$

The requirement that $\gamma = z_2 - z_1 > 0$ shows that $\delta > 1/2$; the further requirement that the upper and lower bounds on $a(x)$ be met, here appearing in the form that $\alpha < z_1 < z_2 < 1$, leads to the final range on δ for solutions for case (i) to exist:

$$\delta > \tilde{\delta} = \frac{1}{2}\max\left[\frac{\bar{v} + \alpha^2}{\bar{v} - \alpha^2}, \frac{1 + \bar{v}}{1 - \bar{v}}\right] > \frac{1}{2} \tag{33}$$

The first of these governs when the area at the lower end $x = 1$ of the rod is reduced to a_1 while that at the upper end $x = 0$ is still below a_2; the second controls when the latter bound is first attained. The two bounds are equal when $\bar{v} = \alpha$; that is, when $\bar{A}$ equals the geometric mean $\sqrt{A_1 A_2}$ of the upper and lower section size bounds. Thus case (i) gives solutions for all $\delta > \tilde{\delta}$, giving way to case (iii) when $\bar{v} > \alpha$ and δ drops below

$$\delta^{\text{iii}}_{\max} = \frac{1}{2}\frac{(1+\bar{v})}{(1-\bar{v})}; \tag{34}$$

it changes to case (v) when $\bar{v} < \alpha$ and δ drops below

$$\delta^{\text{v}}_{\max} = \frac{1}{2}\frac{(\bar{v}+\alpha^2)}{(\bar{v}-\alpha^2)} \tag{35}$$

and it passes to case (vii) when

$$\bar{v} = \alpha, \quad \delta = \frac{1}{2}\frac{1+\alpha}{1-\alpha} \tag{36}$$

at the common point in Fig. 6 where cases (i)–(iii)–(v)–(vii) meet.

For case (xii), the value of $z_1 + z_2$ is given immediately by the fixed volume equation. The other equation then can be solved for $z_2 - z_1$ resulting in

$$\begin{aligned} z_1 &= -\frac{2(1-\alpha)\bar{v}}{1-\bar{v}} - \frac{4(1-\alpha)\delta}{\bar{v}(1-\bar{v})}, \\ z_2 &= -\frac{2(1-\alpha)\bar{v}}{1-\bar{v}} + \frac{4(1-\alpha)\delta}{\bar{v}(1-\bar{v})}. \end{aligned} \tag{37}$$

This solution holds as δ increases until the $a = a_2$ region next to $x = 1$ "disappears"; that happens when $z_2 = -\alpha$ at

$$\delta^{\text{xii}}_{\max} = \frac{\bar{v}[(2-\alpha)\bar{v} - \alpha]}{4(1-\alpha)}. \tag{38}$$

This is the equation of the boundary between cases (xii) and (ix) in Fig. 6. Its graph is a piece of parabola opening upward to the right of the minimum point of the parabola. It starts at zero at the point $\bar{v} = \alpha/(2-\alpha)$ where cases (xi), (xii), and (ix) (and (x) also for negative δ) come together, and rises monotonically to $\delta = 1/2$ as $\bar{v}$ approaches 1.

The equations of the other boundaries in Fig. 6 are routinely established by comparing the equations for each type at the z_1 or z_2 value where the cases meet. Cases (ix) and (vii) meet when their common $z_2 = 1$, and lead to another parabola for δ as a function of $\bar{v}$:

$$\delta^{\text{ix}}_{\max} = \frac{(1-\alpha)(1+\alpha)^2 - 2(1+\alpha+2\alpha^2)\bar{v} + 2(5-\alpha)\bar{v}^2}{2(1-\alpha)(3+\alpha)^2} \tag{39}$$

One may check that this goes through the common (v)–(vii)–(ix)–(xi) point at its left end and goes to $\delta = 1/2$ at $\bar{v} = 1$.

The (iii)–(vii) boundary is also a parabola opening upwards:

$$\delta^{\text{vii}}_{\max}\big|_{\bar{v}>\alpha} = \frac{1-2\alpha^2}{4(1-\alpha)^2} + \frac{1}{(1-\alpha)^2}\left[\bar{v} - \left(\alpha + \frac{1}{2}\right)\right]^2 \tag{40}$$

This starts at the common point for (i)–(iii)–(v)–(vii) at $\bar{v} = \alpha$, $\delta = (1+\alpha)/2(1-\alpha)$ and goes to $\delta = 1/2$ also at $\bar{v} = 1$. It has a minimum value given by the first term at $\alpha + 0.5$ when $\alpha < 0.5$; this minimum value itself has a minimum of 1/4 in $0 < \alpha < 0.5$ so that the (iii)–(vii) boundary is always above 1/4.

The boundary between (i) and (iii) was given earlier in the solution for case (i), also holding only for $\bar{v} > \alpha$. The (i)–(v) boundary was found there also, holding for $\bar{v} < \alpha$.

The next boundary down is that between (v) and (vii). It is a segment of a parabola opening downward to the left of where the maximum point would be, rising monotonically from the (v)–(vii)–(ix)–(xi) common point to the (i)–(iii)–(v)–(vii) point:

$$\delta^{\text{vii}}_{\max}\big|_{\bar{v}<\alpha} = \frac{2-\alpha^2}{4(1-\alpha)^2} - \frac{1}{\alpha^2(1-\alpha)^2}\left[\bar{v} - \left(\alpha + \frac{\alpha^2}{2}\right)\right]^2 \tag{41}$$

The two remaining boundaries are the (v)–(xi) and the (ix)–(xi) ones. These are best given parametrically. For (v)–(xi)

$$\delta^{\text{xi}}_{\max}\big|_{\bar{v}<\frac{\alpha(1+\alpha)}{3-\alpha}} = \frac{1-z_1^2}{2(3-z_1)^2}, \quad \bar{v}z_1^2 + (\alpha^2 - 3\bar{v})z_1 + \alpha^2 = 0. \tag{42}$$

Here in the solution for z_1 the minus sign is chosen before the square root in the quadratic equation formula so that z_1 will run from 1 to α as $\bar{v}$ goes from α^2 to $\alpha(1+\alpha)/(3-\alpha)$.

For the (ix)–(xi) boundary,

$$\delta^{\text{xi}}_{\max}\big|_{\bar{v}>\frac{\alpha(1+\alpha)}{3-\alpha}} = \frac{z_2^2-\alpha^2}{2(4-\alpha-z_2)^2}, \quad \bar{v}z_2^2 + (\alpha - (4-\alpha)\bar{v})z_2 + \alpha^2 = 0. \tag{43}$$

For this δ to be 0 where $\bar{v} = \alpha/(2-\alpha)$ at the common (ix)–(xi)–(xii) point, where $z_2 = \alpha$, the minus sign must be chosen in the quadratic equation formula. At the other end where $\bar{v} = \alpha(1+\alpha)/(3-\alpha)$, the value of z_2 will be $\alpha(3-\alpha)/(1+\alpha)$ and δ will be the desired

$$\delta^{\text{xi}}_{\max}\big|_{\bar{v}=\frac{\alpha(1+\alpha)}{3-\alpha}} = \frac{1-\alpha^2}{2(3-\alpha)^2}. \tag{44}$$

This completes the description of Fig. 6.

7. The solution process for cases (iii)–(xi)

Solution of the (z_1, z_2) equations for the remaining five cases must be numerical for different choices of the parameters. These solutions can be guided by study of the equations and their interpretation as curves in the (z_1, z_2) plane.

The case (iii) Eq. (26) repeated here are

$$\begin{cases} z_2[(1-\bar{v})z_1 + \bar{v}z_2 - 2\alpha\bar{v}] + \alpha^2 = 0 \\ 2\delta[z_2 - z_1 - 2\alpha]^2 = z_1^2 + z_2^2 - 2\alpha^2 \end{cases}$$

The first equation is that of a family of hyperbolas in the (z_1, z_2) plane with asymptotes readily apparent from the equation. The branch used is that for $z_2 > 0$, $z_1 < 0$ since the admissible region is $\alpha < z_2 < 1$, $z_1 < -\rho$. All the hyperbolas go through the corner $z_2 = \alpha$, $z_1 = -\alpha$ of this region, but this is not a possible solution point since it lies on the line $\gamma = z_2 - z_1 - 2\alpha = 0$. The lowest value of $\bar{v}$ at which a hyperbola touches the admissible region again is $\bar{v} = \alpha$, at the corner $z_2 = 1$, $z_1 = -\alpha$. For $\bar{v}$ increasing, the hyperbola will cut through the admissible region from somewhere on the line $z_1 = -\alpha$ to the line $z_2 = 1$. Solutions will lie on this arc of hyperbola where the graph of the other equation cuts it.

The second equation is a conic section with discriminant $4(4\delta - 1)$. Earlier the boundary in Fig. 6 between the (iii) and (vii) regions was shown to have a minimum value greater than 1/4, so this discriminant is positive and the curve is also a hyperbola. Its equation can be written as

$$(4\delta - 1)\left[z_2 - z_1 - \frac{8\alpha\delta}{4\delta - 1}\right]^2 - (z_2 + z_1)^2 = \frac{4\alpha^2}{4\delta - 1} \tag{45}$$

which is that for a rectangular hyperbola relative to axes rotated 45° from the (z_1, z_2) axes and with origin at $z_2 = -z_1 = 4\alpha\delta/(4\delta - 1)$.

For case (v), the $\bar{v}$ equation is again a hyperbola. The δ equation has discriminant $-4(4\delta + 1)$ and so graphs as an ellipse for $\delta > 0$. The asymptotes for the hyperbolas can be seen from the first of Eq. (27)

$$\begin{cases} z_1[\bar{v}z_1 - (\bar{v} - \alpha^2)z_2 - 2\bar{v}] + \alpha^2 = 0 \\ 2\delta[z_2 - z_1 + 2]^2 = 2 - z_1^2 - z_2^2 \end{cases} \tag{46}$$

All hyperbolas pass through the point $z_2 = -1$, $z_1 = 1$ at the corner of the admissible region $-1 < z_2 < 1$, $\alpha < z_1 < 1$ and again this does not represent a solution since it lies on the $\gamma = 0$ line. No solutions exist for $\bar{v} > \alpha$, the first intersection with the admissible region occurring at $\bar{v} = \alpha$, at the corner $z_2 = -1$, $z_1 = \alpha$. For each lower value of $\bar{v}$, there is a hyperbola entering the region at $z_2 = -1$ and leaving on $z_1 = \alpha$, with an arc of the hyperbola in the admissible region where solutions can lie.

The other equation can be rewritten as

$$(4\delta - 1)\left[z_2 - z_1 + \frac{8\delta + 1}{4\delta + 1}\right]^2 + (z_2 + z_1)^2 = \frac{1}{4\delta + 1} \tag{47}$$

This an ellipse with principal axes rotated 45° from (z_1, z_2) and center shifted.

Case (vii) apparently should allow for a closed-form solution in the same way that (xii) did, since the equations of (28) for (vii) are also those of a straight line and a hyperbola:

$$\begin{cases} (1 - \bar{v})z_1 + (\bar{v} - \alpha^2)z_2 + 2(1 - \alpha)\bar{v} = 0 \\ 2\delta[z_2 - z_1 + 2(1 - \alpha)]^2 = 2(1 - \alpha^2) + z_1^2 - z_2^2 \end{cases}$$

with the discriminant of the hyperbola equal to 4 for all δ. However, the algebra is messy and there are actually three subcases for the hyperbola family. Its equation can be rewritten in a form making the asymptotes apparent:

$$(z_2 - z_1)[(2\delta + 1)z_2 - (2\delta - 1)z_1 + 8(1 - \alpha)\delta] = 2(1 - \alpha^2) - 8(1 - \alpha)^2\delta \tag{48}$$

While the asymptote $z_2 = z_1$ holds for all the hyperbolas, the other changes the sign of its slope as δ changes from values above 1/2 to values below 1/2. Its intercept with the z_2 axis is at $-8(1 - \alpha)\delta/(2\delta + 1)$, which is always less than $-2(1 - \alpha)$.

Case (ix) has Eq. (30) represented by hyperbolas for the $\bar{v}$ equation but by either hyperbolas or ellipse for the δ equation:

$$\begin{cases} z_2[-(1 - \bar{v})z_1 + \bar{v}z_2 - 2(2 - \alpha)\bar{v}] + \alpha^2 = 0 \\ 2\delta[2(2 - \alpha) - z_1 - z_2]^2 = z_1^2 + z_2^2 - 2\alpha^2 \end{cases} \tag{49}$$

The discriminant of the second equation is $4(4\delta - 1)$ and so $\delta > 1/4$ corresponds to hyperbolas and $\delta < 1/4$ to ellipses.

Case (xi) has its δ equation represented by a family of hyperbolas again, with the discriminant of the hyperbolas equal to 4 for all δ:

$$\begin{cases} \bar{v}z_1z_2(z_1 + z_2) - 4\bar{v}z_1z_2 + \alpha^2(z_1 + z_2) = 0 \\ 2\delta[4 - z_1 - z_2]^2 = z_2^2 - z_1^2 \end{cases} \tag{50}$$

The asymptotes for the hyperbolas are easily identified by rewriting the latter equation:

$$(z_1 + z_2)[(2\delta + 1)z_1 + (2\delta - 1)z_2 - 16\delta] + 32\delta = 0 \tag{51}$$

The branch of the hyperbola which opens towards $+z_2$ is to be selected. Note that the maximum value of δ which is attained at the (v)–(vii)–(ix)–(xi) common point is less than 1/2, so that the coefficient $(2\delta - 1)$ of z_2 in the brackets is negative.

The other equation represents a cubic curve and other than the symmetry of interchange in z_1 and z_2 there seems to be nothing obvious about the nature of the curve. It is possible by introducing new variables $s = z_1 z_2$, $t = z_1 + z_2$ to show that the curve is a hyperbola in the (s, t) plane:

$$\left(s + \frac{\alpha^2}{\bar{v}}\right)(t - 4) + 4\frac{\alpha^2}{\bar{v}} = 0 \tag{52}$$

It is also possible to show that the pair of equations is equivalent to a single explicit fifth-order polynomial in $t = z_1 + z_2$ by solving the δ equation for $z_2 - z_1$ and substituting in the other:

$$t^3[\bar{v}t(4 - t) - 4\alpha^2] = 4\bar{v}\delta^2(4 - t)^3 \tag{53}$$

8. Formulas for computing the value of the potential energy

The potential energy functional can be rewritten as

$$\pi = \int_0^1 a(x)\lambda(x)\,\mathrm{d}x = \int_0^1 a(x)[\lambda_0 + (\lambda(x) - \lambda_0)]\,\mathrm{d}x = \lambda_0\bar{a} + \int_0^1 a(x)(\lambda(x) - \lambda_0)\,\mathrm{d}x \tag{54}$$

We see that the value of π for comparison theorem designs differs from $\lambda_0\bar{a}$ by terms arising from intervals where $a(x)$ equals one of its bounds, positive where $a = a_2$ and $\lambda \geqslant \lambda_0$ and negative where $a = a_1$ and $\lambda \geqslant \lambda_0$

For cases (i) and (ii) where $\lambda = \lambda_0$ everywhere, $\pi = \lambda_0\bar{a}$ and is easily calculated from the exact solution. For (i)

$$\pi^{(i)} = \frac{\bar{a}}{8}(2\delta - 1)^2 \tag{55}$$

holding for those values of δ for which (i) gives the optimal solution. If this is compared to the energy for the constant-section solution

$$\bar{\pi} = \frac{\bar{a}}{8}\left[(2\delta - 1)^2 - \frac{4}{3}\right] \tag{56}$$

we see that $\pi^{(i)}$ is always greater than $\bar{\pi}$ by the constant amount $\bar{a}/6$.

For the remainder of the cases, numerical results must be calculated. Formulas for these depending on the y or z variables can be found, with the additional integral giving terms that are at most cubic in the variables. Formulas for the cases when $\delta > 0$ follow.

Case (iii):

$$\begin{aligned} 2\gamma^3\pi &= \frac{4}{3}a_1\alpha + \frac{1}{3}a_2 z_1^3 - a_1\frac{(2\alpha^2 - z_1^2)}{z_2}, \\ \gamma &= z_2 - z_1 - 2\alpha \end{aligned} \tag{57}$$

Case (v):

$$\begin{aligned} 2\gamma^3\pi &= -\frac{4}{3}a_1 + a_1 z_1 - 2a_1 z_2 + a_1 z_1^2 z_2 + \frac{2}{3}a_1 z_2^3, \\ \gamma &= z_2 - z_1 + 2 \end{aligned} \tag{58}$$

Case (vii):

$$2\gamma^3\pi = -\frac{4}{3}a_1(1-\alpha) - 2a_1z_2(1-\alpha^2) + \frac{1}{3}a_2z_1^3 + a_1z_1^2z_2 + \frac{2}{3}a_1z_2^3,$$
$$\gamma = z_2 - z_1 + 2(1-\alpha) \tag{59}$$

Case (ix):

$$2\gamma^3\pi = -\frac{4}{3}a_1(2-\alpha) + \frac{1}{3}a_2z_1^3 + a_1\frac{(2\alpha^2 - z_1^2)}{z_2},$$
$$\gamma = 2(2-\alpha) - (z_1 + z_2) \tag{60}$$

Case (xi):

$$2\gamma^3\pi = -\frac{8}{3}a_1 + a_1z_1 + a_1\frac{z_1^2}{z_2},$$
$$\gamma = 4 - (z_1 + z_2) \tag{61}$$

Case (xii):

$$2\gamma^3\pi = -\frac{8}{3}a_1(1-\alpha) + \frac{1}{3}a_2z_1^3 + a_2z_1^2z_2 - \frac{2}{3}a_2z_2^3,$$
$$\gamma = 4(1-\alpha) - (z_1 + z_2) \tag{62}$$

9. The rod with lower end free

The solution for the hanging rod with free lower end can be deduced from the work above. Since the load and strain at $x = 1$ or $y = y_2$ is zero, we must have in the master function the value y_2 coming at the midpoint of the minimum section interval, i.e., $y_2 = 0$. From Fig. 5, we see that the optimal solution must correspond to case (v) or case (vii), with the former holding for smaller values of the prescribed volume and the latter for larger values.

By substituting $y_2 = 0$ in the (v) and (vii) equations, solving the volume equation for y_1 and then using the δ equation, we can find the δ vs. $\bar{v}$ behavior. For case (v), when $\bar{v}$ lies in the range $(\alpha^2, \bar{v}^* = \alpha/(2-\alpha))$, this corresponds to y_1 in $(-1, -(2-\alpha))$ with

$$y_1 = -\left[1 + \sqrt{\left(1 - \frac{\bar{v}}{\alpha^2}\right)}\right] \tag{63a}$$

and

$$\delta = \frac{2 - (2 + y_1)^2}{2y_1^2} \tag{63b}$$

Thus δ will drop from 1/2 (the value of the end deflection for a constant-section rod with free lower end) to

$$\delta^* = \frac{2 - \alpha^2}{2(2-\alpha)^2} \tag{64}$$

in the case (v) region.

For case (vii), where $\bar{v}^* \leqslant \bar{v} \leqslant 1$, we find

$$\delta = \frac{1}{2}\bar{v}^2 + \frac{1}{4}\frac{(1+\alpha)}{(1-\alpha)}(1-\bar{v})^2. \tag{65}$$

This δ equals δ^* at $\bar{v} = \bar{v}^*$ and 1/2 at $\bar{v} = 1$. It has a minimum value

$$\hat{\delta} = \frac{1+\alpha}{2(3-\alpha)} < \frac{1}{2} \tag{66}$$

at $\bar{v} = \hat{\bar{v}} = (1+\alpha)/(3-\alpha)$, where $\bar{v}^* \leqslant \hat{\bar{v}} \leqslant 1$.

10. The "max–min" problem with no upper bound on area

If the only inequality constraint on $a(x)$ is a lower bound constraint $a_1 \leqslant a(x)$, then the "max–min" problem becomes somewhat simpler. Only five cases instead of 12 are involved in the answer, corresponding to cases (i), (ii), (v), (vi), and (xi) above. The master functions have three parts instead of five and are easily constructed:

$$a^*(y) = \begin{cases} \dfrac{a_1}{(2+y)^2}, & -2 < y \leqslant -1 \\ a_1, & -1 \leqslant y \leqslant 1 \\ \dfrac{a_1}{(2-y)^2}, & 1 \leqslant y < 2 \end{cases} \tag{67}$$

$$u^*(y) + \lambda_0^* = \begin{cases} \dfrac{1}{2}(2+y)^2 \\ 1 - \dfrac{1}{2}y^2 \\ \dfrac{1}{2}(2-y)^2 \end{cases} \tag{68}$$

$$\lambda^*(y) - \lambda_0^* = \begin{cases} 0 \\ y^2 - 1 \\ 0 \end{cases} \tag{69}$$

The equations for the five cases are formulated as before by imposing the displacement boundary conditions and volume constraint on functions derived from the master functions using the x to y mapping. Call the five cases here (A), (B), (C), (D), (E), with (A) and (C) corresponding to positive δ values, (B) and (D) to the "reflected" compressive values, and (E) to values of δ straddling zero. It suffices to set and solve the equations for (A), (C), and (E), with (A) solvable in closed form. The equations are written using the ratio $\hat{v} = \bar{a}/a_1 > 1$.

Case (A): $-2 < y_1 < y_2 < -1, \gamma = y_2 - y_1$:

$$\begin{cases} 2\gamma^2\lambda_0 = (2+y_1)^2 \\ 2\gamma^2\delta = (2+y_2)^2 - (2+y_1)^2 \\ \gamma\hat{v} = \dfrac{\gamma}{(2+y_1)(2+y_2)} \end{cases} \tag{70}$$

Case (C): $-2 < y_1 < -1 < y_2 < 1, \gamma = y_2 - y_1$:

$$\begin{cases} 2\gamma^2\lambda_0 = (2+y_1)^2 \\ 2\gamma^2\delta = 2 - y_2^2 - (2+y_1)^2 \\ \gamma\hat{v} = y_2 + \dfrac{1}{2+y_1} \end{cases} \tag{71}$$

Case (E): $-2 < y_1 < -1, 1 < y_2 < 2, \gamma = y_2 - y_1$:

$$\begin{cases} 2\gamma^2\lambda_0 = (2+y_1)^2 \\ 2\gamma^2\delta = (2-y_2)^2 - (2+y_1)^2 \\ \gamma\hat{v} = \dfrac{1}{2-y_2} + \dfrac{1}{2+y_1} \end{cases} \tag{72}$$

Case (A) is solved in the same way as case (i) above, giving

$$\begin{cases} z_1 = 2 + y_1 = \dfrac{1}{\sqrt{\hat{v}}}\sqrt{\dfrac{2\delta-1}{2\delta+1}} \\ z_2 = 2 + y_2 = \dfrac{1}{\sqrt{\hat{v}}}\sqrt{\dfrac{2\delta+1}{2\delta-1}} \end{cases} \tag{73}$$

This solution holds only for

$$\delta > \delta_{\max}^{(C)} = \frac{1}{2}\left(\frac{\hat{v}+1}{\hat{v}-1}\right), \quad \hat{v} > 1 \tag{74}$$

Case (C) holds for each $\hat{v}$ for δ values between $\delta_{\max}^{(C)}$ and $\delta_{\max}^{(E)}$. The value of $\delta_{\max}^{(E)}$ is found by comparing the equations for (C) and (E) at their common value of $y_2 = 1$. A parametric equation for $\delta_{\max}^{(E)}$ is

$$\delta_{\max}^{(E)} = -\frac{(3+y_1)(1+y_1)}{2(1-y_1)^2} \tag{75a}$$

with

$$y_1 = -\left[\frac{1+\hat{v}+\sqrt{(9\hat{v}-1)(\hat{v}-1)}}{2\hat{v}}\right]. \tag{75b}$$

If one were to plot a figure like that of Fig. 6 here, one would find a three-region figure with (A) at top, (C) in the central part, and (E) at the bottom with the boundary $\delta_{\max}^{(C)}(\hat{v})$ between (A) and (C) obtained as though the common point in Fig. 6 between (i)–(iii)–(v)–(vii) moved down and to the right, approaching the value $\delta = 1/2$ monotonically from above as $\hat{v}$ becomes unbounded. Similarly, the boundary $\delta_{\max}^{(E)}(\hat{v})$ between (C) and (E) is obtained by having the common point between (v)–(vii)–(ix)–(xi) move up and to the right, approaching the value $\delta = 1/18$ monotonically from below as $\hat{v}$ becomes unbounded.

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